

# KAWAHARA-BURGERS EQUATION ON A STRIP

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**ABSTRACT.** An initial-boundary value problem for the 2D Kawahara-Burgers equation posed on a channel-type strip was considered. The existence and uniqueness results for regular and weak solutions in weighted spaces as well as exponential decay of small solutions without restrictions on the width of a strip were proven both for regular solutions in an elevated norm and for weak solutions in the  $L^2$ -norm.

## 1. INTRODUCTION

We are concerned with an initial-boundary value problem (IBVP) for the two-dimensional Kawahara-Burgers (KB) equation

$$u_t + u_x - u_{xx} + uu_x + u_{xxx} + u_{xyy} - \partial_x^5 u = 0 \quad (1.1)$$

posed on a strip modeling an infinite channel  $\{(x, y) \in \mathbb{R}^2 : x \in \mathbb{R}, y \in (0, B), B > 0\}$ . This equation is a two-dimensional analog of the Kawahara type equation

$$u_t - \partial_x^5 u + F(u, u_x, u_{xx}, u_{xxx}) = 0 \quad (1.2)$$

which includes dissipation and dispersion and has been studied intensively last years due to its applications in Mechanics and Physics [1, 3, 4, 5, 7, 6, 8, 18, 25].

Equations (1.1) and (1.2) are typical examples of so-called dispersive equations which attract considerable attention of both pure and applied mathematicians in the past decades. The theory of the Cauchy problem for (1.2) and other dispersive equations like the KdV equation has been extensively studied and is considerably advanced today [1, 4, 5, 6, 7, 8, 19, 20, 18, 22, 23, 25, 37, 40]. Results on IBVPs for one-dimensional dispersive equations both in bounded and unbounded

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domains may be found in [5, 6, 9, 10, 24, 28, 32]. It was shown in [9, 10, 27, 29, 30, 33] that the KdV and Kawahara equations have an implicit internal dissipation. This allowed the proof of exponential decay of small solutions in bounded domains without adding any artificial damping term. Later, this effect has been proven for a wide class of dispersive equations of any odd order with one space variable [16].

On the other hand, it has been shown in [39] that control of the linear KdV equation with the transport term  $u_x$  may fail for critical domains, but it is possible to eliminate the term  $u_x$  by simple scaling when the KdV and Kawahara equations are posed on the whole line. The same is true also for (1.1) posed on a strip ( $y \in (0, B)$ ,  $x \in \mathbb{R}$ ,  $t > 0$ ) [31].

Recently, interest on dispersive equations became to be extended to multi-dimensional models such as Kadomtsev-Petviashvili (KP), Zakharov-Kuznetsov (ZK) equations [42] and dispersive equations of higher orders [12]. As far as the ZK equation and its generalizations are concerned, the results on IVPs can be found in [13, 17, 35, 36, 38] and IBVPs were studied in [2, 14, 15, 29, 34, 41]. It was shown that IBVP for the ZK equation posed on a half-strip unbounded in  $x$  direction with the Dirichlet conditions on the boundaries possesses regular solutions which decay exponentially as  $t \rightarrow \infty$  provided initial data are sufficiently small and the width of a half-strip is not too large [30, 34]. The similar result was established for the 2D Kawahara equation posed on a half-strip [29]. This means that multi-dimensional dispersive equation may create an internal dissipative mechanism for some types of IBVPs.

The goal of our note is to prove that the KB equation on a strip also may create a dissipative effect without adding any artificial damping. We must mention that IBVP for the ZK equation on a strip ( $x \in (0, 1)$ ,  $y \in \mathbb{R}$ ) has been studied in [11, 41] and IBVPs on a strip ( $y \in (0, L)$ ,  $x \in \mathbb{R}$ ) for the ZK equation and Zakharov-Kuznetsov-Burgers equation were considered in [2, 31] and for the ZK equation with some internal damping in [15]. In the domain ( $y \in (0, B)$ ,  $x \in \mathbb{R}$ ,  $t > 0$ ), the term  $u_x$  in (1.1) can be scaled out by a simple change of variables. Nevertheless, it can not be safely ignored for problems posed both on finite and semi-infinite intervals as well as on infinite in  $y$  direction bands without changes in the original domain [11, 39].

The main results of our paper are the existence and uniqueness of regular and weak global-in-time solutions for (1.1) posed on a strip with the Dirichlet boundary conditions and the exponential decay rate of these solutions as well as continuous dependence on initial data. To explore dissipativity of the term  $u_{xyy}$ , we used exponential weight  $e^{2bx}$

which implied to define solutions of (1.1) as the product

$$e^{bx}[u_t - u_{xx} + uu_x + u_{xxx} + u_{xyy} - \partial_x^5 u] = 0 \quad \text{in } L^2(\mathcal{S}).$$

We must mention that this idea has been proposed yearlier in [19].

The paper has the following structure. Section 1 is Introduction. Section 2 contains formulation of the problem. In Section 3, we prove global existence and uniqueness theorems for regular solutions in some weighted spaces and continuous dependence on initial data. Surprisingly, we did not succeed to prove global existence for all positive weights  $e^{2bx}$  as in [30, 34] and imposed a restriction  $6 - 40b^2 \geq 0$ . In Section 4, we prove exponential decay of small regular solutions in an elevated norm. In Section 5, we prove the existence, uniqueness and continuous dependence on initial data for weak solutions as well as the exponential decay rate of the  $L^2(\mathcal{S})$ -norm for small solutions without limitations on the width of the strip.

## 2. PROBLEM AND PRELIMINARIES

Let  $B, T, r$  be finite positive numbers. Define  $\mathcal{S} = \{(x, y) \in \mathbb{R}^2 : x \in \mathbb{R}, y \in (0, B)\}$ ;  $\mathcal{S}_r = \{(x, y) \in \mathbb{R}^2 : x \in (-r, +\infty), y \in (0, B)\}$  and  $\mathcal{S}_T = \mathcal{S} \times (0, T)$ .

Hereafter subscripts  $u_x, u_{xy}$ , etc. denote the partial derivatives, as well as  $\partial_x$  or  $\partial_{xy}^2$  when it is convenient. Operators  $\nabla$  and  $\Delta$  are the gradient and Laplacian acting over  $\mathcal{S}$ . By  $(\cdot, \cdot)$  and  $\|\cdot\|$  we denote the inner product and the norm in  $L^2(\mathcal{S})$ , and  $\|\cdot\|_{H^k}$  stands for norms in the  $L^2$ -based Sobolev spaces. We will use also the spaces  $H^s \cap L_b^2$ , where  $L_b^2 = L^2(e^{2bx} dx)$ , see [19].

Consider the following IBVP:

$$Lu \equiv u_t - u_{xx} + uu_x + u_{xxx} + u_{xyy} - \partial_x^5 u = 0, \quad \text{in } \mathcal{S}_T; \quad (2.1)$$

$$u(x, 0, t) = u(x, B, t) = 0, \quad x \in \mathbb{R}, \quad t > 0; \quad (2.2)$$

$$u(x, y, 0) = u_0(x, y), \quad (x, y) \in \mathcal{S}. \quad (2.3)$$

## 3. EXISTENCE OF REGULAR SOLUTIONS

**Approximate solutions.** We will construct solutions to (2.1)-(2.3) by the Faedo-Galerkin method: let  $w_j(y)$  be orthonormal in  $L^2(\mathcal{S})$  eigenfunctions of the following Dirichlet problem:

$$w_{jyy} + \lambda_j w_j = 0, \quad y \in (0, B); \quad (3.1)$$

$$w_j(0) = w_j(B) = 0. \quad (3.2)$$

Define approximate solutions of (2.1)-(2.3) as follows:

$$u^N(x, y, t) = \sum_{j=1}^N w_j(y) g_j(x, t), \quad (3.3)$$

where  $g_j(x, t)$  are solutions to the following Cauchy problem for the system of  $N$  generalized Kawahara equations:

$$\begin{aligned} \frac{\partial}{\partial t} g_j(x, t) + \frac{\partial^3}{\partial x^3} g_j(x, t) - \frac{\partial^2}{\partial x^2} g_j(x, t) - \lambda_j \frac{\partial}{\partial x} g_j(x, t) \\ - \frac{\partial^5}{\partial x^5} g_j(x, t) + \int_0^B u^N(x, y, t) u_x^N(x, y, t) w_j(y) dy = 0, \end{aligned} \quad (3.4)$$

$$g_j(x, 0) = \int_0^B w_j(y) u_0(x, y) dy, \quad j = 1, \dots, N. \quad (3.5)$$

It can be shown that for  $g_j(x, 0) \in H^s$ ,  $s \geq 5$ , the Cauchy problem (3.4)-(3.5) has a unique regular solution  $g_j \in L^\infty(0, T; H^s(\mathcal{S}) \cap L_b^2(\mathcal{S})) \cap L^2(0, T; H^{s+2}(\mathcal{S}) \cap L_b^2(\mathcal{S}))$  [1, 18, 19, 37]. To prove the existence of global solutions for (2.1)-(2.3), we need uniform in  $N$  global in  $t$  estimates of approximate solutions  $u^N(x, y, t)$ .

**Estimate I.** Multiply the  $j$ -th equation of (3.4) by  $g_j$ , sum up over  $j = 1, \dots, N$  and integrate the result with respect to  $x$  over  $\mathbb{R}$  to obtain

$$\frac{d}{dt} \|u^N\|^2(t) + 2 \|u_x^N\|^2(t) = 0$$

which implies

$$\|u^N\|^2(t) + 2 \int_0^t \|u_x^N\|^2(s) ds = \|u_0^N\|^2 \quad \forall t \in (0, T). \quad (3.6)$$

It follows from here that for  $N$  sufficiently large and  $\forall t > 0$

$$\|u^N\|^2(t) + 2 \int_0^t \|u_x^N\|^2(s) ds = \|u^N\|^2(0) \leq 2 \|u_0\|^2. \quad (3.7)$$

In our calculations we will drop the index  $N$  where it is not ambiguous.

**Estimate II.** For some positive  $b$ , multiply the  $j$ -th equation of (3.4) by  $e^{2bx} g_j$ , sum up over  $j = 1, \dots, N$  and integrate the result with respect to  $x$  over  $\mathbb{R}$ . Dropping the index  $N$ , we get

$$\begin{aligned} \frac{d}{dt} (e^{2bx}, u^2)(t) + (2 + 6b - 40b^3) (e^{2bx}, u_x^2)(t) + 2b (e^{2bx}, u_y^2)(t) \\ + 10b (e^{2bx}, u_{xx}^2)(t) - \frac{4b}{3} (e^{2bx}, u^3)(t) \\ - (4b^2 + 8b^3 - 32b^5) (e^{2bx}, u^2)(t) = 0. \end{aligned} \quad (3.8)$$

**Proposition 3.1.** *Let  $b \in (0, \frac{\sqrt{0.6}}{2}]$ , then*

$$6b - 40b^3 \geq 0. \quad (3.9)$$

The proof is obvious.

In our calculations, we will frequently use the following multiplicative inequalities [26]:

**Proposition 3.2.** *i) For all  $u \in H^1(\mathbb{R}^2)$*

$$\|u\|_{L^4(\mathbb{R}^2)}^2 \leq 2\|u\|_{L^2(\mathbb{R}^2)}\|\nabla u\|_{L^2(\mathbb{R}^2)}. \quad (3.10)$$

*ii) For all  $u \in H^1(D)$*

$$\|u\|_{L^4(D)}^2 \leq C_D\|u\|_{L^2(D)}\|u\|_{H^1(D)}, \quad (3.11)$$

where the constant  $C_D$  depends on a way of continuation of  $u \in H^1(D)$  as  $\tilde{u}(\mathbb{R}^2)$  such that  $\tilde{u}(D) = u(D)$ .

Extending  $u^N(x, y, t)$  for a fixed  $t$  into the exterior of  $\mathcal{S}$  by 0 and exploiting (3.10), we find

$$\frac{4b}{3}(e^{2bx}u^3)(t) \leq b(e^{2bx}, u_y^2)(t) + 2b(e^{2bx}, u_x^2)(t) + 2(b^3 + \frac{8b}{9}\|u_0^N\|^2)(e^{2bx}, u^2)(t).$$

Substituting this into (3.8), we come to the inequality

$$\begin{aligned} & \frac{d}{dt}(e^{2bx}, u^2)(t) + (e^{2bx}, u_x^2 + u_y^2 + u_{xx}^2)(t) \\ & \leq C(b)(1 + \|u_0\|^2)(e^{2bx}, u^2)(t). \end{aligned} \quad (3.12)$$

By the Gronwall lemma,

$$(e^{2bx}, u^2)(t) \leq C(b, T, \|u_0\|)(e^{2bx}, u_0^2).$$

Returning to (3.12) gives

$$\begin{aligned} & (e^{2bx}, |u^N|^2)(t) + \int_0^t (e^{2bx}, |\nabla u^N|^2 + |u_{xx}^N|^2)(\tau) d\tau \\ & \leq C(b, T, \|u_0\|)(e^{2bx}, u_0^2) \quad \forall t \in (0, T). \end{aligned} \quad (3.13)$$

It follows from this estimate and (3.6) that uniformly in  $N$  and for any  $r > 0$  and  $t \in (0, T)$

$$\begin{aligned} & \|u^N\|^2(t) + \int_0^t \int_0^B \int_{-r}^{+\infty} [|\nabla u^N|^2 + |u_{xx}^N|^2] dx dy ds \\ & \leq \mathbb{C}(r, b, T, \|u_0\|)(e^{2bx}, u_0^2), \end{aligned} \quad (3.14)$$

where  $\mathbb{C}$  does not depend on  $N$ .

Estimates (3.13), (3.14) make it possible to prove the existence of a weak solution to (2.1)-(2.3) passing to the limit in (3.4) as  $N \rightarrow \infty$ . For details of passing to the limit in the nonlinear term see [19].

We will need the following lemma :

**Lemma 3.3.** *Let  $u(x, y) : \mathcal{S} \rightarrow \mathbb{R}$  be such that*

$$\int_{\mathcal{S}} e^{2bx} [u^2(x, y) + |\nabla u(x, y)|^2 + u_{xy}^2(x, y)] dx dy < \infty$$

*and for all  $x \in \mathbb{R}$  there is some  $y_0 \in [0, B]$  such that  $u(x, y_0) = 0$ . Then*

$$\begin{aligned} \sup_{\mathcal{S}} |e^{bx} u(x, y, t)|^2 &\leq \delta(1 + 2b^2)(e^{2bx}, u_y^2)(t) + 2\delta(e^{2bx}, u_{xy}^2)(t) \\ &+ \frac{2\delta_1}{\delta}(e^{2bx}, u_x^2)(t) + \frac{1}{\delta} \left[ \frac{1}{\delta_1} + 2\delta_1 b^2 \right] (e^{2bx}, u^2)(t), \end{aligned} \quad (3.15)$$

*where  $\delta, \delta_1$  are arbitrary positive numbers.*

*Proof.* Denote  $v = e^{bx} u$ . Then simple calculations give

$$\sup_{\mathcal{S}} v^2(x, y, t) \leq \delta[\|v_y\|^2(t) + \|v_{xy}\|^2(t)] + \frac{1}{\delta}[\|v_x\|^2(t) + \|v\|^2(t)].$$

Returning to the function  $u(x, y, t)$ , we prove Lemma 3.3  $\square$

**Estimate III.** Multiplying the  $j$ -th equation of (3.4) by  $-(e^{2bx} g_{jx})_x$ , and dropping the index  $N$ , we come to the equality

$$\begin{aligned} \frac{d}{dt}(e^{2bx}, u_x^2)(t) &+ (2 + 6b - 40b^3)(e^{2bx}, u_{xx}^2)(t) + 2b(e^{2bx}, u_{xy}^2)(t) \\ &+ 10b(e^{2bx}, u_{xxx}^2)(t) - (4b^2 + 8b^3 - 32b^5)(e^{2bx}, u_x^2)(t) \\ &+ (e^{2bx}, u_x^3)(t) - 2b(e^{2bx}, uu_x^2)(t) = 0. \end{aligned} \quad (3.16)$$

Making use of Proposition 3.2, we estimate

$$\begin{aligned} I_1 &= (e^{2bx}, u_x^3)(t) \leq \|u_x\|(t) \|e^{bx} u_x\|_{L^4(\mathcal{S})}^2(t) \\ &\leq 2\|u_x\|(t) \|e^{bx} u_x\|(t) \|\nabla(e^{bx} u_x)\|(t) \\ &\leq \delta(e^{2bx}, 2u_{xx}^2 + u_{xy}^2)(t) + 2\left[\delta b^2 + \frac{\|u_x\|^2(t)}{2\delta}\right] (e^{2bx}, u_x^2)(t). \end{aligned}$$

Similarly,

$$\begin{aligned} I_2 &= 2b(e^{2bx}, uu_x^2)(t) \leq \delta(e^{2bx}, 2u_{xx}^2 + u_{xy}^2)(t) \\ &+ \left[2b^2\delta + \frac{4b^2}{\delta}\|u_0\|^2(t)\right] (e^{2bx}, u_x^2)(t). \end{aligned}$$

Substituting  $I_1, I_2$  into (3.16) with  $2\delta = b$ , we obtain for  $\forall t \in (0, T)$  :

$$\begin{aligned} & (e^{2bx}, |u_x^N|^2)(t) + \int_0^t (e^{2bx}, |\nabla u_x^N|^2 + |u_{xxx}^N|^2)(s) ds \\ & \leq C(b, T, \|u_0\|)(e^{2bx}, u_{0x}^2). \end{aligned} \quad (3.17)$$

**Estimate IV.** Multiplying the  $j$ -th equation of (3.4) by  $-2(e^{2bx} \lambda_j g_j)$ , and dropping the index  $N$ , we come to the equality

$$\begin{aligned} & \frac{d}{dt}(e^{2bx}, u_y^2)(t) + (2 + 6b - 40b^3)(e^{2bx}, u_{xy}^2)(t) + 2b(e^{2bx}, u_{yy}^2)(t) \\ & + 10b(e^{2bx}, u_{xxy}^2)(t) - (4b^2 + 8b^3 - 32b^5)(e^{2bx}, u_y^2)(t) \\ & + 2(1 - b)(e^{2bx}, u_x u_y^2)(t) = 0. \end{aligned} \quad (3.18)$$

Making use of Proposition 3.2, we estimate

$$\begin{aligned} I &= 2(1 - b)(e^{2bx}, u_x u_y^2)(t) \\ &\leq 2C_D(1 + b)\|u_x\|(t)\|e^{bx} u_y\|(t)\|(e^{bx} u_y)\|_{H^1(S)}(t) \\ &\leq \delta(e^{2bx}, 2u_{xy}^2 + u_{yy}^2)(t) + [2\delta(1 + b^2) \\ &\quad + \frac{C_D^2(1 + b)^2\|u_x\|^2(t)}{\delta}](e^{2bx}, u_y^2)(t). \end{aligned}$$

Taking  $\delta = b$ , we transform (3.18) into the inequality

$$\begin{aligned} & \frac{d}{dt}(e^{2bx}, u_y^2)(t) + (e^{2bx}, u_{xy}^2 + u_{yy}^2 + u_{xxy}^2)(t) \\ & \leq C(b)[1 + \|u_x\|(t)^2](e^{2bx}, u_y^2)(t). \end{aligned}$$

Making use of (3.7) and the Gronwall lemma, we get  $\forall t \in (0, T)$  :

$$\begin{aligned} & (e^{2bx}, |u_y^N|^2)(t) + \int_0^t (e^{2bx}, |u_{yy}^N|^2 + |u_{xy}^N|^2 + |u_{xxy}^N|^2)(s) ds \\ & \leq C(b, T, \|u_0\|)(e^{2bx}, u_{0y}^2). \end{aligned} \quad (3.19)$$

This and (3.17) give for  $\forall t \in (0, T)$ :

$$\begin{aligned} & (e^{2bx}, |\nabla u^N|^2)(t) + \int_0^t (e^{2bx}, |\nabla u_x^N|^2 + |\nabla u_{xx}^N|^2 + |u_{yy}^N|^2)(s) ds \\ & \leq C(b, T, \|u_0\|)(e^{2bx}, |\nabla u_0|^2) \end{aligned} \quad (3.20)$$

which imply that for all finite  $r > 0$  and all  $t \in (0, T)$

$$\|u^N\|(t)_{H^1(S_r)} \leq C(r, b, T, \|u_0\|)(e^{2bx}, |\nabla u_0|^2). \quad (3.21)$$

**Estimate V.** Multiplying the  $j$ -th equation of (3.4) by  $(e^{2bx}g_{jxx})_{xx}$ , and dropping the index  $N$ , we come to the equality

$$\begin{aligned} & \frac{d}{dt}(e^{2bx}, u_{xx}^2)(t) + (2 + 6b - 40b^3)(e^{2bx}, u_{xxx}^2)(t) + 2b(e^{2bx}, u_{xxy}^2)(t) \\ & + 10b(e^{2bx}, u_{xxx}^2)(t) - (4b^2 + 8b^3 - 32b^5)(e^{2bx}, u_{xx}^2)(t) \\ & - 2b(e^{2bx}, uu_{xx}^2)(t) + 5(e^{2bx}u_x, u_{xx}^2)(t) = 0. \end{aligned} \quad (3.22)$$

Using (3.10), we estimate

$$\begin{aligned} I &= -2b(e^{2bx}, uu_{xx}^2)(t) + 5(e^{2bx}u_x, u_{xx}^2)(t) \\ &\leq 2\delta(e^{2bx}, 2u_{xxx}^2 + u_{xxy}^2)(t) + \left[4b^2\delta + \frac{25}{\delta}\|u_x\|(t)^2\right. \\ &\quad \left.+ \frac{4b^2}{\delta}\|u\|^2(t)\right](e^{2bx}, u_{xx}^2)(t). \end{aligned}$$

Taking  $2\delta = b$  and substituting  $I$  into (3.22), we obtain

$$\begin{aligned} & \frac{d}{dt}(e^{2bx}, u_{xx}^2)(t) + (e^{2bx}, u_{xxx}^2 + u_{xxy}^2 + u_{xxx}^2)(t) \\ & \leq C(b)[1 + \|u_x\|^2(t) + \|u\|^2(t)](e^{2bx}, u_{xx}^2)(t). \end{aligned}$$

Making use of (3.7), we find

$$\begin{aligned} & (e^{2bx}, |u_{xx}^N|^2)(t) + \int_0^t (e^{2bx}, |\nabla u_{xx}^N|^2 + |u_{xxx}^N|^2)(s) ds \\ & \leq C(b, T, \|u_0\|)(e^{2bx}, u_{0xx}^2) \quad \forall t \in (0, T). \end{aligned} \quad (3.23)$$

**Estimate VI.** Differentiate (3.4) by  $t$  and multiply the result by  $e^{2bx}g_{jt}$  to obtain

$$\begin{aligned} & \frac{d}{dt}(e^{2bx}, u_t^2)(t) + (2 + 6b - 40b^3)(e^{2bx}, u_{xt}^2)(t) + 2b(e^{2bx}, u_{ty}^2)(t) \\ & + 10b(e^{2bx}, u_{txx}^2)(t) - (4b^2 + 8b^3 - 32b^5)(e^{2bx}, u_t^2)(t) \\ & + (2 - 2b)(e^{2bx}u_x, u_t^2)(t) = 0. \end{aligned} \quad (3.24)$$

Making use of (3.10), we estimate

$$\begin{aligned} I &= (2 - 2b)(e^{2bx}u_x, u_t^2)(t) \leq 2(2 + 2b)\|u_x\|(t)\|e^{bx}u_t\|(t)\|\nabla(e^{bx}u_t)\|(t) \\ & \leq \delta(e^{2bx}, 2u_{xt}^2 + u_{ty}^2)(t) + \left[2b^2\delta + \frac{(2 + 2b)^2\|u_x\|(t)^2}{\delta}\right](e^{2bx}, u_t^2)(t). \end{aligned}$$

Taking  $\delta = b$  and substituting  $I$  into (3.24), we get



$$\begin{aligned} & \frac{d}{dt}(e^{2bx}, u_t^2)(t) + (e^{2bx}, u_{xt}^2 + u_{ty}^2 + u_{txx}^2)(t) \\ & \leq C(b)[1 + \|u_x\|(t)^2](e^{2bx}, u_t^2)(t). \end{aligned}$$

This implies  $\forall t \in (0, T)$ :

$$\begin{aligned} & (e^{2bx}, |u_t^N|^2)(t) + \int_0^t (e^{2bx}, |\nabla u_s^N|^2 + |u_{sxx}^N|^2)(s) ds \\ & \leq C(b, T, \|u_0\|)(e^{2bx}, u_t^2)(0) \leq C(b, T, \|u_0\|)J_0, \end{aligned} \quad (3.25)$$

where

$$J_0 = \|u_0\|^2 + (e^{2bx}, u_0^2 + |\nabla u_0|^2 + |\nabla u_{0x}|^2 + u_0^2 u_{0x}^2 + |\Delta u_{0x}|^2 + |\partial_x^5 u_0|^2).$$

**Estimate VII.** Multiplying the  $j$ -th equation of (3.4) by  $-e^{2bx} g_{jx}$  and dropping the index  $N$ , we come to the equality

$$\begin{aligned} & (e^{2bx}, [u_{xy}^2 + u_{xxx}^2])(t) = (e^{2bx}, uu_x^2)(t) + (e^{2bx}, u_t, u_x)(t) \\ & + (8b^2 - 1)(e^{2bx}, u_{xx}^2)(t) + (b + 2b^2 - 8b^4)(e^{2bx}, u_x^2)(t). \end{aligned} \quad (3.26)$$

Using (3.10), we estimate

$$I = (e^{2bx}, uu_x^2)(t) \leq \delta(e^{2bx}, 2u_{xx}^2 + u_{xy}^2)(t) + \left[2b^2\delta + \frac{\|u_0\|^2}{\delta}\right](e^{2bx}, u_x^2)(t).$$

Taking  $2\delta = 1$ , using (3.17)-(3.25) and substituting  $I$  into (3.26), we get

$$(e^{2bx}, u_{xxx}^{N^2} + u_{xy}^{N^2})(t) \leq C(b, T, \|u_0\|)J_0 \quad \forall t \in (0, T). \quad (3.27)$$

**Estimate VIII.**

Multiplying the  $j$ -th equation of (3.4) by  $e^{2bx} g_{jxxx}$ , we come, dropping the index  $N$ , to the equality

$$\begin{aligned} & (e^{2bx}, u_{xxy}^2 + u_{xxx}^2)(t) = -(e^{2bx}, [u_t - u_{xx}], u_{xxx})(t) - (e^{2bx}, uu_x, u_{xxx})(t) \\ & + 2b^2(e^{2bx}, u_{xy}^2)(t) + (2b^2 - 1)(e^{2bx}, u_{xxx}^2)(t). \end{aligned} \quad (3.28)$$

Using Lemma 3.3 and (3.7), we estimate

$$\begin{aligned} I & = (e^{2bx}, uu_x, u_{xxx})(t) \leq \|u\|(t) \sup_S |e^{bx} u_x(x, y, t)| \|e^{bx} u_{xxx}\|(t) \\ & \leq \frac{\|u_0\|^2}{2}(e^{2bx}, u_{xxx}^2)(t) + \frac{1}{2} \left[ \frac{1}{\delta}(1 + 2b^2)(e^{2bx}, u_x^2)(t) \right. \\ & \quad \left. + \frac{2}{\delta}(e^{2bx}, u_{xx}^2)(t) + \delta(1 + 2b^2)(e^{2bx}, u_{xy}^2)(t) + 2\delta(e^{2bx}, u_{xxy}^2)(t) \right]. \end{aligned} \quad (3.29)$$

Taking  $\delta$  sufficiently small, positive and substituting  $I$  into (3.28), we find

$$(e^{2bx}, |\nabla u_{xx}^N|^2 + |\partial_x^4 u^N|^2)(t) \leq C(b, T, \|u_0\|)J_0 \quad \forall t \in (0, T). \quad (3.30)$$

Consequently, it follows from the equalities:

$$-(e^{2bx}[u_t^N - u_{xx}^N + u_{xxx}^N + u_{xyy}^N + u^N u_x^N - \partial_x^5 u^N], u_{yy}^N)(t) = 0,$$

$$-(e^{2bx}[u_t^N - u_{xx}^N + u_{xxx}^N + u_{xyy}^N + u^N u_x^N - \partial_x^5 u^N], \partial_x^5 u^N)(t) = 0$$

and

$$(e^{2bx}[u_t^N - u_{xx}^N + u_{xxx}^N + u_{xyy}^N + u^N u_x^N - \partial_x^5 u^N], u_{xyy}^N)(t) = 0$$

that

$$\begin{aligned} & (e^{2bx}, |u_{yy}^N|^2 + |u_{xyy}^N|^2 + |\partial_x^5 u^N|^2 + |u_{xxxxy}^N|^2)(t) \\ & \leq C(b, T, \|u_0\|)J_0 \quad \forall t \in (0, T). \end{aligned} \quad (3.31)$$

Jointly, estimates (3.17), (3.19), (3.23), (3.27), (3.30), (3.31) read

$$\begin{aligned} & (e^{2bx}, |u^N|^2 + |\nabla u^N|^2 + |\nabla u_x^N|^2 + |\nabla u_y^N|^2 + |\nabla u_{xx}^N|^2 + |\Delta u_x^N|^2 \\ & + |\nabla u_{xxx}^N|^2 + |\partial_x^5 u^N|^2)(t) \leq C(b, T, \|u_0\|)J_0 \quad \forall t \in (0, T). \end{aligned} \quad (3.32)$$

In other words,

$$\begin{aligned} e^{bx} u^N, \quad e^{bx} u_x^N & \in L^\infty(0, T; H^2(\mathcal{S})) \\ \nabla u_{xxx}^N, \quad \partial_x^5 u^N & \in L^\infty(0, T; L^2(\mathcal{S})) \end{aligned} \quad (3.33)$$

and these inclusions are uniform in  $N$ .

**Estimate IX.** Multiplying the  $j$ -th equation of (3.4) by  $e^{2bx} \lambda_j^2 g_j$ , we come, dropping the index  $N$ , to the equality

$$\begin{aligned} & b(e^{2bx}, 5u_{xxyy}^2 + u_{yyy}^2)(t) = (2b^2 + 4b^3 - 16b^5)(e^{2bx}, u_{yy}^2)(t) \\ & + (20b^3 - 3b - 1)(e^{2bx}, u_{xyy}^2)(t) + (e^{2bx}, u_{ty}, u_{yyy})(t) + (e^{2bx}, uu_{xy}, u_{yyy})(t) \\ & + (e^{2bx}, u_y u_x, u_{yyy})(t). \end{aligned} \quad (3.34)$$

We estimate

$$\begin{aligned}
I_1 &= -(e^{2bx}, u_{ty}, u_{yyy})(t) \leq \frac{\epsilon}{2}(e^{2bx}, u_{yyy}^2)(t) + \frac{1}{2\epsilon}(e^{2bx}, u_{yt}^2)(t), \\
I_2 &= (e^{2bx} u_y u_x, u_{yyy})(t) \leq \|u_x\|(t) \|e^{bx} u_{yyy}\|(t) \sup_S |e^{bx} u_y(x, y, t)| \\
&\leq \frac{\epsilon}{2}(e^{2bx}, u_{yyy}^2)(t) + \frac{\|u_x\|(t)^2}{2\epsilon} [(1 + 2b^2)(e^{2bx}, u_y^2)(t) \\
&\quad + 2(e^{2bx}, u_{xy}^2)(t) + (1 + 2b^2)(e^{2bx}, u_{yy}^2)(t) + 2(e^{2bx}, u_{xyy}^2)(t)], \\
I_3 &= (e^{2bx} u u_{xy}, u_{yyy})(t) \leq \|u\|(t) \|e^{bx} u_{yyy}\|(t) \sup_S |e^{bx} u_{xy}(x, y, t)| \\
&\leq \frac{\|u_0\|^2 \epsilon_1}{2}(e^{2bx}, u_{yyy}^2)(t) + \frac{1}{2\epsilon_1} [2\delta(e^{2bx}, u_{xxyy}^2)(t) \\
&\quad + \frac{2}{\delta}(e^{2bx}, u_{xxy}^2)(t) + \delta(1 + 2b^2)(e^{2bx}, u_{xyy}^2)(t) \\
&\quad + \frac{1}{\delta}(1 + 2b^2)(e^{2bx}, u_{xy}^2)(t)].
\end{aligned}$$

Choosing  $\epsilon$ ,  $\epsilon_1$ ,  $\delta$  sufficiently small, positive, after integration, we transform (3.34) into the form

$$\int_0^T (e^{2bx}, [|u_{xxyy}^N|^2 + |u_{yyy}^N|^2])(t) dt \leq C(b, T, \|u_0\|) J_0. \quad (3.35)$$

Acting similarly, we get from the scalar product

$$(e^{2bx} [u_t^N - u_{xx}^N + u_{xxx}^N + u_{xyy}^N + u^N u_x^N - \partial_x^5 u^N], u_{xyyyy}^N)(t) = 0$$

the estimate

$$\int_0^T (e^{2bx}, |u_{xyyy}^N|^2 + |\partial_x^3 u_{yy}^N|^2)(t) dt \leq C(b, T, \|u_0\|) J_0. \quad (3.36)$$

Estimates (3.32), (3.33), (3.35), (3.36) guarantee that

$$e^{bx} u^N, \quad e^{bx} u_x^N \in L^\infty(0, T; H^2(\mathcal{S}) \cap L^2(0, T; H^3(\mathcal{S})) \quad (3.37)$$

and these inclusions do not depend on  $N$ . Independence of Estimates (3.7), (3.37) of  $N$  allow us to pass to the limit in (3.4) and to prove the following result:

**Theorem 3.4.** *Let  $u_0(x, y) : \mathbb{R}^2 \rightarrow \mathbb{R}$  be such that  $u_0(x, 0) = u_0(x, B) = 0$  and for some  $b > 0$  satisfying (3.9)*

$$J_0 = \|u_0\|^2 + (e^{2bx}, u_0^2 + |\nabla u_0|^2 + |\nabla u_{0x}|^2 + u_0^2 u_{0x}^2 + |\Delta u_{0x}|^2 + |\partial_x^5 u_0|^2) < \infty.$$

Then there exists a regular solution to (2.1)-(2.3)  $u(x, y, t)$  :

$$\begin{aligned} u &\in L^\infty(0, T; L^2(\mathcal{S})), \quad u_x \in L^2(0, T; L^2(\mathcal{S})) \\ e^{bx}u, e^{bx}u_x &\in L^\infty(0, T; H^2(\mathcal{S})) \cap L^2(0, T; H^3(\mathcal{S})) \\ e^{bx}u_t &\in L^\infty(0, T; (L^2(\mathcal{S}))) \cap L^2(0, T; H^1(\mathcal{S})), \\ e^{bx}u_{xxt} &\in L^2(0, T; L^2(\mathcal{S})), \quad e^{bx}\partial_x^5 u \in L^2(0, T; H^1(\mathcal{S})) \end{aligned}$$

which for a.e.  $t \in (0, T)$  satisfies the identity

$$(e^{bx}[u_t - u_{xx} + u_{xxx} + uu_x + u_{xyy} - \partial_x^5 u], \phi(x, y))(t) = 0, \quad (3.38)$$

where  $\phi(x, y)$  is an arbitrary function from  $L^2(\mathcal{S})$ .

*Proof.* Rewrite (3.4) in the form

$$\begin{aligned} (e^{bx}[u_t^N - u_{xx}^N + u^N u_x^N + u_{xxx}^N + u_{xyy}^N \\ - \partial_x^5 u^N], \Phi^N(y)\Psi(x))(t) = 0, \end{aligned} \quad (3.39)$$

where  $\Phi^N(y)$  is an arbitrary function from the set of linear combinations  $\sum_{i=1}^N \alpha_i w_i(y)$  and  $\Psi(x)$  is an arbitrary function from  $H^1(\mathbb{R})$ . Taking into account estimates (3.7), (3.37) and fixing  $\Phi^N$ , we can easily pass to the limit as  $N \rightarrow \infty$  in linear terms of (3.39). To pass to the limit in the nonlinear term, we must use (3.21) and repeat arguments of [19]. Since linear combinations  $[\sum_{i=1}^N \alpha_i w_i(y)]\Psi(x)$  are dense in  $L^2(\mathcal{S})$ , we come to (3.38). This proves the existence of regular solutions to (2.1)-(2.3).  $\square$

**Remark 1.** Estimates (3.7), (3.37) are valid also for the limit function  $u(x, y, t)$  and (3.7) obtains its sharp form:

$$\|u\|(t)^2 + 2 \int_0^t \|u_x\|^2(s) ds = \|u_0\|^2 \quad \forall t \in (0, T). \quad (3.40)$$

### Uniqueness of a regular solution.

**Theorem 3.5.** A regular solution from Theorem 3.4 is uniquely defined.

*Proof.* Let  $u_1, u_2$  be two distinct regular solutions of (2.1)-(2.3), then  $z = u_1 - u_2$  satisfies the following initial-boundary value problem:

$$z_t - z_{xx} + z_{xxx} + z_{xyy} - \partial_x^5 z + \frac{1}{2}(u_1^2 - u_2^2)_x = 0 \text{ in } \mathcal{S}_T, \quad (3.41)$$

$$z(x, 0, t) = z(x, B, t) = 0, \quad x \in \mathbb{R}, \quad t > 0, \quad (3.42)$$

$$z(x, y, 0) = 0. \quad (x, y) \in \mathcal{S}. \quad (3.43)$$

Multiplying (3.41) by  $2e^{bx}z$ , we get

$$\begin{aligned} & \frac{d}{dt}(e^{2bx}, z^2)(t) + (2 + 6b - 40b^3)(e^{2bx}, z_x^2)(t) + 10b(e^{2bx}, z_{xx}^2)(t) \\ & - (4b^2 + 8b^3 - 32b^5)(e^{2bx}, z^2)(t) + (e^{2bx}[u_{1x} + u_{2x}], z^2)(t) \\ & + 2b(e^{2bx}, z_y^2)(t) - b(e^{2bx}(u_1 + u_2), z^2)(t) = 0. \end{aligned} \quad (3.44)$$

We estimate

$$\begin{aligned} I_1 &= (e^{2bx}(u_{1x} + u_{2x}), z^2)(t) \leq \|u_{1x} + u_{2x}\|(t) \|e^{bx}z\|_{L^4(\mathcal{S})}^2(t) \\ &\leq 2\|u_{1x} + u_{2x}\|(t) \|e^{bx}z\|(t) \|\nabla(e^{bx}z)\|(t) \leq \delta(e^{2bx}, [2z_x^2 + z_y^2])(t) \\ &+ [2b^2\delta + \frac{2}{\delta}(\|u_{1x}\|^2(t) + \|u_{2x}\|^2(t))](e^{2bx}, z^2)(t), \\ I_2 &= b(e^{2bx}(u_1 + u_2), z^2)(t) \leq b\|u_1 + u_2\|(t) \|e^{bx}z\|_{L^4(\mathcal{S})}^2(t) \\ &\leq 2b\|u_1 + u_2\|(t) \|e^{bx}z\|(t) \|\nabla(e^{bx}z)\|(t) \\ &\leq \delta(e^{2bx}, 2z_x^2 + z_y^2)(t) + [2b^2\delta + \frac{2b^2}{\delta}(\|u_1\|^2(t) + \|u_2\|^2(t))](e^{2bx}, z^2)(t). \end{aligned}$$

Substituting  $I_1, I_2$  into (3.44) and taking  $\delta > 0$  sufficiently small, we find

$$\begin{aligned} & \frac{d}{dt}(e^{2bx}, z^2)(t) \leq C(b)[1 + \|u_1\|^2(t) \\ & + \|u_2\|^2(t) + \|u_{1x}\|^2(t) + \|u_{2x}\|^2(t)](e^{2bx}, z^2)(t). \end{aligned} \quad (3.45)$$

Since

$$u_i \in L^\infty(0, T; L^2(\mathcal{S})), \quad u_{ix} \in L^2(0, T; L^2(\mathcal{S})) \quad i = 1, 2,$$

then by the Gronwall lemma,

$$(e^{2bx}, z^2)(t) = 0 \quad \forall t \in (0, T).$$

Hence,  $u_1 = u_2$  a.e. in  $\mathcal{S}_T$ .  $\square$

**Remark 2.** Changing initial condition (3.43) for  $z(x, y, 0) = z_0(x, y) \neq 0$ , and repeating the proof of Theorem 3.4, we obtain from (3.45) that

$$(e^{2bx}, z^2)(t) \leq C(b, T, \|u_0\|)(e^{2bx}, z_0^2) \quad \forall t \in (0, T).$$

This means continuous dependence of regular solutions on initial data.

#### 4. DECAY OF REGULAR SOLUTIONS

In this section we will prove exponential decay of regular solutions in an elevated weighted norm. We start with Theorem ?? which is crucial for the main result.

**Theorem 4.1.** *Let  $b \in (0, b_0)$ ,  $\|u_0\| \leq \frac{3\pi}{8B}$  and  $u(x, y, t)$  be a regular solution of (2.1)-(2.3). Then for all finite  $B > 0$  the following inequalities are true:*

$$\|e^{bx}u\|^2(t) \leq e^{-\chi t} \|e^{bx}u_0\|^2(0), \quad (4.1)$$

$$\int_0^t e^{\chi s} (e^{2bx}, |\nabla u|^2)(s) ds \leq C(b, \|u_0\|)(1+t)(e^{2bx}, u_0^2), \quad (4.2)$$

where

$$\chi = \frac{b_0\pi^2}{4B^2}, \quad b_0 = \min\left(\frac{\sqrt{0,6}}{2}, \quad \frac{1}{5}[-1 + \sqrt{1 + \frac{5\pi^2}{4B^2}}]\right).$$

*Proof.* Multiplying (2.1) by  $2e^{2bx}u$ , we get the equality

$$\begin{aligned} & \frac{d}{dt}(e^{2bx}, u^2)(t) + (2 + 6b - 40b^3)(e^{2bx}, u_x^2)(t) + 10b(e^{2bx}, u_{xx}^2)(t) \\ & + 2b(e^{2bx}, u_y^2)(t) - \frac{4b}{3}(e^{2bx}, u^3)(t) \\ & - (4b^2 + 8b^3 - 32b^5)(e^{2bx}, u^2)(t) = 0. \end{aligned} \quad (4.3)$$

Taking into account (3.10), we estimate

$$\begin{aligned} I &= \frac{4b}{3}(e^{2bx}, u^3)(t) \leq b(e^{2bx}, u_y^2 + 2u_x^2 + 2b^2u^2)(t) \\ &+ \frac{16b}{9}\|u_0\|^2(e^{2bx}, u^2)(t). \end{aligned}$$

The following proposition is principal for our proof.

**Proposition 4.2.**

$$\int_{\mathbb{R}} \int_0^B e^{2bx} u^2(x, y, t) dy dx \leq \frac{B^2}{\pi^2} \int_{\mathbb{R}} \int_0^B e^{2bx} u_y^2(x, y, t) dy dx. \quad (4.4)$$

*Proof.* Since  $u(x, 0, t) = u(x, B, t) = 0$ , fixing  $(x, t)$ , we can use with respect to  $y$  the following Steklov inequality: if  $f(y) \in H_0^1(0, \pi)$  then

$$\int_0^\pi f^2(y) dy \leq \int_0^\pi |f_y(y)|^2 dy.$$

After a corresponding process of scaling we prove Proposition 4.2.  $\square$

Making use of (4.4) and substituting  $I$  into (4.3), we come to the following inequality:

$$\begin{aligned} & \frac{d}{dt}(e^{2bx}, u^2)(t) + (e^{2bx}, u_x^2)(t) \\ & + \left[\frac{b\pi^2}{B^2} - 4b^2 - 10b^3 - \frac{16b}{9}\|u_0\|^2\right](e^{2bx}, u^2)(t) \leq 0 \end{aligned}$$

which can be rewritten as

$$\frac{d}{dt}(e^{2bx}, u^2)(t) + \chi(e^{2bx}, u^2)(t) \leq 0, \quad (4.5)$$

where

$$\chi = b\left[\frac{\pi^2}{B^2} - 4b - 10b^2 - \frac{16\|u_0\|^2}{9}\right].$$

Since we need  $\chi > 0$ , define

$$4b + 10b^2 = \gamma \frac{\pi^2}{B^2}, \quad \frac{16\|u_0\|^2}{9} = (1 - \gamma)^2 \frac{\pi^2}{B^2}, \quad (4.6)$$

where  $\gamma \in (0, 1)$ . It implies  $\chi = bA(\gamma)\frac{\pi^2}{B^2}$  with  $A(\gamma) = \gamma(1 - \gamma)$ . It is easy to see that

$$\sup_{\gamma \in (0,1)} A(\gamma) = A\left(\frac{1}{2}\right) = \frac{1}{4}.$$

Solving (4.6), we find

$$b = \frac{1}{5}\left[-1 + \sqrt{1 + \frac{5\pi^2}{4B^2}}\right], \quad \|u_0\| \leq \frac{3\pi}{8B}, \quad \chi = b\frac{\pi^2}{4B^2},$$

and from (4.5) we get

$$(e^{2bx}, u^2)(t) \leq e^{-\chi t}(e^{2bx}, |u_0|^2).$$

The last inequality implies (4.1). □

To prove (4.2), we return to (4.3) and multiply it by  $e^{\chi t}$  to obtain

$$\begin{aligned} \frac{d}{dt}[e^{\chi t}(e^{2bx}, u^2)(t)] + e^{\chi t}[(2 + 6b^2 - 40b^3)(e^{2bx}, u_x^2)(t) + 2b(e^{2bx}, u_y^2)(t) \\ + 10b(e^{2bx}, u_{xx}^2)(t)] = \frac{4be^{\chi t}}{3}(e^{2bx}, u^3)(t) + \\ e^{\chi t}(\chi + 4b^2 + 8b^3 - 32b^5)(e^{2bx}, u^2)(t). \end{aligned} \quad (4.7)$$

Acting as above, we find

$$\begin{aligned} I = \frac{4b}{3}(e^{2bx}, u^3)(t) \leq b(e^{2bx}, 2u_x^2 + u_y^2)(t) + 10b(e^{2bx}, u_{xx}^2)(t) \\ + (2b^2 + \frac{16b\|u_0\|^2}{9})(e^{2bx}, u^2)(t). \end{aligned} \quad (4.8)$$

Substituting this into (4.7), we get

$$\begin{aligned} \frac{d}{dt}[e^{\chi t}(e^{2bx}, u^2)(t)] + e^{\chi t}(e^{2bx}, |\nabla u|^2)(t) + 10be^{\chi t}(e^{2bx}, u_{xx}^2)(t) \\ \leq C(b, \|u_0\|)e^{\chi t}(e^{2bx}, u^2)(t). \end{aligned} \quad (4.9)$$

Integrating and (4.1) imply

$$\begin{aligned} e^{\chi t}(e^{2bx}, u^2)(t) + \int_0^t e^{\chi s}(e^{2bx}, |\nabla u|^2 + u_{xx}^2)(s) ds \\ \leq C(b, \|u_0\|)(1+t)(e^{2bx}, u_0^2). \end{aligned} \quad (4.10)$$

The proof of Theorem 4.1 is complete.

Observe that differently from [29, 30, 34], we do not have any restrictions on the width of a strip  $B$ .

The main result of this section is the following assertion.

**Theorem 4.3.** *Let all the conditions of Theorem 4.1 be fulfilled. Then regular solutions of (2.1)-(2.3) satisfy the following inequality:*

$$\begin{aligned} (e^{2bx}, u^2 + |\nabla u|^2 + u_{xx}^2)(t) \\ \leq C(b, \|u_0\|)(1+t)e^{-\chi t}(e^{2bx}, [u_0^2 + |\nabla u_0|^2 + u_{0xx}^2]) \end{aligned} \quad (4.11)$$

or

$$\|e^{bx}u\|_{H^1(S)}^2(t) + \|u_{xx}\|^2(t) \leq C(b, \|u_0\|)(1+t)e^{-\chi t}(e^{2bx}, u_0^2 + |\nabla u_0|^2 + u_{0xx}^2).$$

*Proof.* We start with the following lemma.

**Lemma 4.4.** *Regular solutions of (2.1)-(2.3) satisfy the following equality:*

$$\begin{aligned} e^{\chi t}(e^{2bx}, |\nabla u|^2 + u_{xx}^2)(t) + 2 \int_0^t e^{\chi s} \{ (1 + 3b - 20b^3)(e^{2bx}, |\nabla u_x|^2 + u_{xxx}^2)(s) \\ + b(e^{2bx}, |\nabla u_y|^2 + u_{xxy}^2)(s) + 5b(e^{2bx}, |\nabla u_{xx}|^2 + |\partial_x^4 u|^2)(s) \} ds = \\ \int_0^t e^{\chi s} (\chi + 4b^2 + 8b^3 - 32b^5)(e^{2bx}, |\nabla u|^2 + u_{xx}^2)(s) ds \\ + \int_0^t e^{\chi s} \{ 4b(e^{2bx}u, u_x^2)(s) + 2(e^{2bx}uu_x, u_{yy} + 4b^2u_{xx} + 4bu_{xxx} + \partial_x^4 u)(s) \\ - (e^{2bx}u^2, u_{xxx} + 2bu_{xx})(s) \} ds + (e^{2bx}, |\nabla u_0|^2 + u_{0xx}^2). \end{aligned} \quad (4.12)$$

*Proof.* First we transform the scalar product

$$\begin{aligned} -2(e^{2bx} [u_t - u_{xx} + u_{xxx} + u_{xyy} + uu_x - \partial_x^5 u], \\ [u_{yy} + u_{xx} - \partial_x^4 u])(t) = 0 \end{aligned} \quad (4.13)$$



into the following equality:

$$\begin{aligned}
& \frac{d}{dt}(e^{2bx}, |\nabla u|^2 + u_{xx}^2)(t) + 2(1 + 3b - 20b^3)(e^{2bx}, |\nabla u_x|^2 + u_{xxx}^2)(t) \\
& + 2b(e^{2bx}, |\nabla u_y|^2 + u_{xxy}^2)(t) + 10b(e^{2bx}, |\nabla u_{xx}|^2 + |\partial_x^4 u|^2)(t) \\
& = 4b^2(1 + 2b - 8b^3)(e^{2bx}, |\nabla u|^2 + u_{xx}^2)(t) + 2(e^{2bx} uu_x, u_{yy} \\
& + 4b^2 u_{xx} + 4bu_{xxx} + \partial_x^4 u)(t) - (e^{2bx} u^2, u_{xxx} + 2bu_{xx})(t) \\
& + 4b(e^{2bx}, uu_x^2)(t). \tag{4.14}
\end{aligned}$$

Multiplying this by  $e^{\chi t}$  and integrating, we prove (4.12).  $\square$

Making use of Lemma 3.3, estimate separate terms in (4.12) as follows:

$$\begin{aligned}
I_1 &= 2(e^{2bx} uu_x, u_{yy} + 4b^2 u_{xx} + 4bu_{xxx} + \partial_x^4 u)(t) \\
&\leq 2\|u\|(t) \sup_S |e^{bx} u_x|(t) \|e^{bx} [u_{yy} + 4b^2 u_{xx} + 4bu_{xxx} + \partial_x^4 u]\|(t) \\
&\leq \epsilon(1 + \|u_0\|^2)(e^{2bx}, u_{yy}^2 + u_{xx}^2 + u_{xxx}^2 + |\partial_x^4 u|^2)(t) \\
&+ \frac{C(b)}{\epsilon} \{2\delta(e^{2bx}, u_{xy}^2 + u_{xxy}^2)(t) + \frac{3}{\delta}(e^{2bx}, u_{xx}^2 + u_x^2)(t)\}. \\
I_2 &= 4b(e^{2bx}, uu_x^2)(s) \leq \delta(e^{2bx}, 2u_{xx}^2 + u_{xy}^2)(s) \\
&+ [2b^2\delta + \frac{16(1+4b)^2\|u_0\|^2}{\delta}](e^{2bx}, u_x^2)(s); \\
I_3 &= (e^{2bx} u^2, u_{xxx} + 2bu_{xx})(t) \leq \|u\|(t) \sup_S |e^{bx} u|(t) \|e^{bx} [u_{xxx} + 2bu_{xx}]\|(t) \\
&\leq \frac{\epsilon}{2}(e^{2bx}, u_{xxx}^2 + u_{xx}^2)(t) + C(b)\|u_0\|^2[\delta(e^{2bx}, u_{xy}^2)(t) \\
&+ \frac{1}{\delta}(2e^{2bx}, |\nabla u|^2 + u^2)(t)].
\end{aligned}$$

Choosing  $\epsilon, \delta$  sufficiently small, substituting  $I_1 - I_3$  into (4.12) and taking into account (4.1), we prove that

$$e^{\chi t}(e^{2bx}, |\nabla u|^2 + u_{xx}^2)(t) \leq C(b, \|u_0\|)(1+t)(e^{2bx}, u_0^2 + |\nabla u_0|^2 + u_{0xx}^2).$$

Adding (4.1), we complete the proof of Theorem 4.3.  $\square$

## 5. WEAK SOLUTIONS

Here we will prove the existence, uniqueness and continuous dependence on initial data as well as exponential decay results for weak solutions of (2.1)-(2.3) when the initial function  $u_0 \in L^2(\mathcal{S})$ .

**Theorem 5.1.** *Let  $u_0 \in L^2(\mathcal{S}) \cap L_b^2(\mathcal{S})$ . Then for all finite positive  $T$  and  $B$  there exists at least one function  $u(x, y, t)$ :*

$$u \in L^\infty(0, T; L^2(\mathcal{S})), \quad u_x \in L^2(0, T; L^2(\mathcal{S}))$$

such that

$$e^{bx}u \in L^\infty(0, T; L^2(\mathcal{S})) \cap L^2(0, T; H^1(\mathcal{S})), \\ e^{bx}u_{xx} \in L^2(0, T; L^2(\mathcal{S}))$$

and the following integral identity takes a place:

$$(e^{bx}u, v)(T) + \int_0^T \left\{ -(e^{bx}u, v_t)(t) - \frac{1}{2}(e^{2bx}u^2, bv + v_x)(t) \right. \\ \left. + (e^{bx}u_{xx}, [v_{xxx} + 3bv_{xx} + (3b^2 - 1)v_x + (b^3 - b - 1)v])(t) \right. \\ \left. + (e^{bx}u_y, bv_y + v_{xy})(t) \right\} dt = (e^{bx}u_0, v(x, y, 0)), \quad (5.1)$$

where  $v \in C^\infty(\mathcal{S}_T)$  is an arbitrary function.

*Proof.* In order to justify our calculations, we must operate with sufficiently smooth solutions  $u^m(x, y, t)$ . With this purpose, we consider first initial functions  $u_{0m}(x, y)$ , which satisfy conditions of Theorem 3.4, and obtain estimates (3.7), (3.21) for functions  $u^m(x, y, t)$ . This allows us to pass to the limit as  $m \rightarrow \infty$  in the following identity:

$$(e^{bx}u^m, v)(T) + \int_0^T \left\{ -(e^{bx}u^m, v_t)(t) - \frac{1}{2}(e^{2bx}|u^m|^2, bv + v_x)(t) \right. \\ \left. + (e^{bx}u_{xx}^m, [v_{xxx} + 3bv_{xx} + (3b^2 - 1)v_x + (b^3 - b - 1)v])(t) \right. \\ \left. + (e^{bx}u_y^m, bv_y + v_{xy})(t) \right\} dt = (e^{bx}u_0^m, v(x, y, 0)), \quad (5.2)$$

and come to (5.1).  $\square$

### Uniqueness of a weak solution.

**Theorem 5.2.** *A weak solution of Theorem 5.1 is uniquely defined.*

*Proof.* Actually, this proof is provided by Theorem 3.5. It is sufficient to approximate the initial function  $u_0 \in L^2(\mathcal{S})$  by regular functions  $u_{0m}$  in the form:

$$\lim_{m \rightarrow \infty} \|u_{0m} - u_0\| = 0,$$

where  $u_{0m}$  satisfies the conditions of Theorem 3.4. This guarantees the existence of the unique regular solution to (2.1)-(2.3) and allows us to repeat all the calculations which have been done during the proof of Theorem 3.5 and to come to the following inequality:

$$\frac{d}{dt}(e^{2bx}, z_m^2)(t) + (e^{2bx}, |\nabla z_m|^2)(t) \\ \leq C(b)[1 + \|u_{1m}\|^2(t) + \|u_{2m}\|^2(t) + \|u_{1xm}\|^2(t) + \|u_{2xm}\|^2(t)](e^{2bx}, z_m^2)(t).$$

By the generalized Gronwall's lemma,

$$(e^{2bx}, z_m^2)(t) \leq \exp\left\{\int_0^t C(b)[1 + \|u_{1m}\|^2(s) + \|u_{2m}\|^2(s) + \|u_{1xm}\|^2(s) + \|u_{2xm}\|^2(s)] ds\right\}(e^{2bx}, z_{0m}^2)(t).$$

Functions  $u_{1m}$  and  $u_{2m}$  for  $m$  sufficiently large satisfy the estimate

$$\|u_{im}\|^2(t) + 2 \int_0^t \|u_{imx}\|^2(s) ds = \|u_{0m}\|^2 \leq 2\|u_0\|^2, \quad i = 1, 2.$$

Hence,

$$\begin{aligned} & \exp\left\{\int_0^t C(b)[1 + \|u_{1m}\|^2(s) + \|u_{2m}\|^2(s) + \|u_{1xm}\|^2(s) \right. \\ & \left. + \|u_{2xm}\|^2(s)] ds\right\} \leq C(b, T, \|u_0\|). \end{aligned} \quad (5.3)$$

Since  $e^{bx}z(x, y, t)$  is a weak limit of regular solutions  $\{e^{bx}z_m(x, y, t)\}$ , then

$$(e^{2bx}, z^2)(t) \leq (e^{2bx}, z_m^2)(t) = 0.$$

This implies  $u_1 \equiv u_2$  a.e. in  $\mathcal{S}_T$ . The proof of Theorem 5.2 is complete.  $\square$

**Remark 3.** Changing initial condition  $z(x, y, 0) \equiv 0$  for  $z(x, y, 0) = z_0(x, y) \neq 0$ , and repeating the proof of Theorem 5.2, we obtain that

$$(e^{2bx}, z^2)(t) \leq C(b, T, \|u_0\|)(e^{2bx}, z_0^2) \quad \forall t \in (0, T).$$

This means continuous dependence of weak solutions on initial data.

### Decay of weak solutions.

**Theorem 5.3.** Let  $b \in (0, b_0)$ ,  $\|u_0\| \leq \frac{3\pi}{16B}$  and  $u(x, y, t)$  be a regular solution of (2.1)-(2.3). Then for all finite  $B > 0$  the following inequality is true:

$$\|e^{bx}u\|^2(t) \leq e^{-\chi t} \|e^{bx}u_0\|^2(0), \quad (5.4)$$

where

$$\chi = \frac{b_0\pi^2}{4B^2}, \quad b_0 = \min\left(\frac{\sqrt{0,6}}{2}, \quad \frac{1}{5}[-1 + \sqrt{1 + \frac{5\pi^2}{4B^2}}]\right).$$

*Proof.* Similarly to the proof of the uniqueness result for a weak solution, we approximate  $u_0 \in L^2(\mathcal{S})$  by sufficiently smooth functions  $u_{0m}$  in order to work with regular solutions. Acting in the same manner as by the proof of Theorem 4.1, we come to the following inequality :

$$\|e^{bx}u_m\|^2(t) \leq e^{-\chi t} \|e^{bx}u_{0m}\|^2(0), \quad (5.5)$$

where

$$\chi = \frac{\pi^2}{20B^2}[-1 + \sqrt{1 + \frac{5\pi^2}{4B^2}}].$$

Since  $u(x, y, t)$  is weak limit of regular solutions  $\{u_m(x, y, t)\}$  then

$$(e^{2bx}, u^2)(t) \leq (e^{2bx}, u_m^2)(t) \leq e^{-\chi t}(e^{2bx}, u_0^2).$$

The proof of Theorem 5.3 is complete.  $\square$

We have in this Theorem a more strict condition  $\|u_0\| \leq \frac{3\pi}{16B}$  instead of  $\|u_0\| \leq \frac{3\pi}{8B}$  in the case of decay for regular solution because for weak solutions we do not have the sharp estimate (3.40), but only (3.7).

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